05 Apr 1995, 1:30 pm - 3:30 pm

Dispersion of Longitudinal Waves Propagating in a Continuum with Randomly Perturbated Parameters

J. Náprstek

Institute of Theoretical and Applied Mechanics, Prague, Czech Republic

Proceedings: Third International Conference on Recent Advances in Geotechnical Earthquake Engineering and Soil Dynamics, April 2-7, 1995, Volume II, St. Louis, Missouri

Dispersion of Longitudinal Waves Propagating in a Continuum with Randomly Perturbated Parameters

Paper No. 10.16

J. Náprstek
Chief Scientific Officer, Institute of Theoretical and Applied Mechanics, Prague, Czech Republic

SYNOPSIS The author investigates the propagation of wave motion in a continuum the material parameters of which are random functions of longitudinal coordinate. The analysis is based on the theory of Markov processes and the subsequent solution of the respective Fokker-Planck-Kolmogorov equation. The fully deterministic response in the excitation point transforms into non-homogeneous random process in the longitudinal coordinate with the growing distance. The quota of the deterministic component in overall response drops until it disappears completely. The model of uncorrelated imperfections (white noise) of the continuum is inacceptable, because it is at variance with the energy equilibrium law. The results are compared with the conclusions resulting from the application of the integral spectral decomposition analysis and the finite element method based on correlation method.

INTRODUCTION

Seismic waves of natural or technological origin propagate through the continuum the physical parameters of which are of the character of random process in space. The wave motion propagating in such a continuum has a number of specific properties which we shall try to demonstrate on a kinematically excited semifinite bar with an axis $x \geq 0$. In dependence on the structure and extent of random deviations from nominal values the absolute value of the measurable deterministic part of the response drops with the distance x from the point of excitation, while its random part quickly increases from zero leaving certain small area around the excitation point. From the initially deterministic process in the point of excitation the response, with the increasing x, is becoming a process with a continuously growing portion of stochastic component.

The effects of random imperfections have been introduced to the response mechanism in previous papers in most varied ways, but always on the basis of the small parameter method - see e.g. Deodasis et al.(1991), or Nakagiri and Hisada (1985). The application of this method is equivalent with the assumtion of statistic independence of response in adjacent points in the generalized meaning. Moreover, it assumes that the deterministic part of the response is dominant for all values of x, while the random part has the meaning merely of a certain not very significant supplement. This approach can be accepted only as a certain approximation in case of bodies of finite dimensions and with sparse spectrum of natural frequencies and deterministic boundary conditions with not very significant increase of the quota of indeterminateness and a drop of determinism of the response with the distance from the point of deterministic excitation. On global scale, however, this model clashes directly with the energy equilibrium law. The significance of the deterministic and stochastic parts of the response for increasing x is getting to reverse from that in the proximity of the point of excitation which is not enabled by the philosophy of the small parameter method. Most difficulties resulting from these procedures can be eliminated by the admission of statistic dependence of imperfections in the longitudinal direction. If one of the two parameters ($\varrho(x)$ - density, $E(x)$ - Young modulus of elesticity) or both (when the term $\partial u(x,t)/\partial x \cdot \partial E(x)/\partial x$ could be neglected) are random variables, it is possible to describe the propagation of the longitudinal wave motion along a semi-infinite bar by the well known equation:

$$\frac{\partial^2 u(x,t)}{\partial x^2} - \frac{1}{c^2(x)}\frac{\partial^2 u(x,t)}{\partial t^2} = 0 \qquad (1)$$

$$c^2(x) = E(x)/\varrho(x) = (\psi^2 + \varphi(x))^{-1} \qquad (2)$$

$$\varrho(x) = \varrho_0 + \varrho_\varepsilon(x) \; ; \; |\varrho_\varepsilon(x)| << \varrho_0$$

where :

$c(x)$ - velocity of longitudinal wave propagation,

$\psi^2 = \mathbf{E}\{c^{-2}(x)\}$ - mathematical mean value of the process $c^{-2}(x)$

$\mathbf{E}\{\cdot\}$ - mathematical mean value operator

$\varphi(x)$ - centered random Gaussian homogeneous process.

In case of harmonic excitation in point $x = 0$ it is possible to write:

$$u(x,t)|_{x=0} = K \cdot e^{i\omega t} \; ; \; u(x,t) = v(x) \cdot e^{i\omega t} \qquad (3)$$

which transforms the problem (1), (2) with respect to (3) into solution of an ordinary differential equation with a randomly variable coefficient:

$$\frac{d^2 v(x)}{dx^2} + \omega^2(\psi^2 + \varphi(x))v(x) = 0 \; ; \; v(0) = K \qquad (4)$$

with the necessity of complying with the Sommerfeld's condition for $x \to \infty$.

Influence of imperfections, or Eq.(4), has been analyzed by Náprstek (1993) using the decomposition in the form of Stiltjes integrals with non-continuous spectral differentials and verified by Náprstek and Frýba (1994) using the finite element method respecting non-zero correlation of imperfections along the x axis. In this short contribution we shall outline the most important steps and the results obtained by the procedure based on Markov processes and compare them with the results of both afore mentioned methods.

UNCORRELATED IMPERFECTIONS

To demonstrate the properties of the uncorrelated imperfections model, we shall introduce the random process $\varphi(x)$ describing them as a Gaussian white noise $\xi(x)$. The corresponding Ito system can be written by means of two phase variables $v_1(x) = v(x), v_2(x) = \dot{v}(x)$ in the form of

$$\left.\begin{aligned} v_1'(x) = & \quad v_2(x) \\ v_2'(x) = & \quad -\omega^2\psi^2 v_1(x) - \omega^2\xi(x)v_1(x) \end{aligned}\right\} \tag{5}$$

from which there follows the Fokker-Planck-Kolmogorov equation for the non-homogeneous cross density of probability $p(v_1, v_2, x)$:

$$\frac{\partial p}{\partial x} = -\frac{\partial(v_2 p)}{\partial v_1} + \omega^2\psi^2\frac{\partial(v_1 p)}{\partial v_2} + \frac{1}{2}\omega^2 s\frac{\partial^2(v_1^2 p)}{\partial v_2^2} \tag{6}$$

where :

s - intensity of white noise $\xi(x)$

In point $x = 0$ the response process is deterministic ; therefore, it holds that

$$p(v_1, v_2, x)|_{x=0} = C\delta(v_1)\delta(v_2) \tag{7}$$

where : $\delta(\cdot)$ - Dirac function.

The classic solution of Eq. (6),(7) does not exist. Therefore, we shall seek a generalized solution in the meaning of stochastic moments - see e.g. Bolotin (1979) or Pugachev and Sinitsyn (1987). We shall multiply Eq. (6) by the factor $v_1^j v_2^k$ and integrate within the whole phase area. After modifications we shall arrive at the system of equations:

$$\frac{d}{dx}U_{j,k} = -\omega^2\psi^2 \cdot kU_{j+1,k-1} + jU_{j-1,k+1} + \frac{s}{2}\omega^2 k(k-1)U_{j+2,k-2} \tag{8}$$

$$U_{j,k} = \iint_{-\infty}^{\infty} p(v_1, v_2, x)v_1^j v_2^k dv_1 dv_2 \quad ; \quad 0 \le j,k < \infty \tag{9}$$

The expression (9) is the statistic moment of the degree $j+k$ according to the phase variables v_1,v_2. Consequently, it also holds that :

$$U_{j,k} = \mathbf{E}\{v_1^j \cdot v_2^k\} \tag{10}$$

The structure of Eq.(8) reveals that a certain degree of moments κ (i.e. for $j + k = \kappa = const.$) it will always disintegrate into separate systems of $\kappa + 1$ equations. That means that the moments of the κ degree influence one another only within this degree. For $\kappa = 1$, i.e. for the mathematical mean values of the quantities v_1,v_2 , we shall obtain :

$$\left.\begin{aligned} U_{1,0}' = & \quad U_{0,1} \\ U_{0,1}' = & \quad -\omega^2\psi^2 U_{1,0} \end{aligned}\right\} \tag{11}$$

from which, by the elimination of $U_{0,1}$, we obtain the equation which formally coincides with Eq. (4) for $\varphi(x) \equiv 0$. The mean value of the deviation of U_{10} or $\mathbf{E}\{v_1(x)\}$, therefore, with the assumption of the delta-correlated process $\varphi(x)$, equals to the solution of the classic problem for ideal parameters of ϱ_0, E_0:

$$U_{10} = K \cdot exp(-i\omega\psi x) \tag{12}$$

For $\kappa = 2$, i.e. for autocorrelations and cross correlation of $v_1(x), v_2(x)$, it follows from Eq.(8) :

$$\left.\begin{aligned} U_{20}' = & \quad 2U_{11} \\ U_{11}' = & \quad -\omega^2\psi^2 U_{20} + U_{02} \\ U_{02}' = & \quad -2\omega^2\psi^2 U_{11} + \omega^2 s U_{20} \end{aligned}\right\} \tag{13}$$

The system of Eq.(13) is homogeneous similarly as all others for $\kappa > 2$. Because the response in point $x = 0$ is deterministic, also the initial conditions for all partial systems are homogeneous for $\kappa > 1$. With regard to the fact that we are seeking the solution within an infinite interval, the solution of the system of Eq.(13) and all others for $\kappa > 2$ can be only trivial.

This results is, obviously, contradictory. It would mean that response is by no means influenced by imperfections. From Eq.(8) we would obtain $|U_{10}| = const.$, meaning that all energy would be permanently concentrated in the deterministic part of the response. The stochastic part of the response would equal zero, and should it not be so, the source of energy producing it would be unclear. Therefore, we can conclude, in accordance with the results of the spectral decomposition method. Náprstek (1993) and the finite element method, Náprstek and Frýba (1994), that the response cannot be modelled as white noise and that the non-zero correlation of $\varphi(x)$ in longitudinal direction must be admitted.

IMPERFECTIONS OF DIFFUSION TYPE

The autocorrelation of imperfections in longitudinal direction can be described in the simplest case by the exponential function. We shall arrive at white noise as input by the "filter" of the first order in the variable x, described by the equation

$$\varphi'(x) + a\varphi(x) = \xi(x) \tag{14}$$

which corresponds with the spectral density of imperfections - see e.g. Bolotin (1993), or Pugachev and Sinitsyn (1987):

$$S_\varphi(\alpha) = \frac{\sigma_\varphi^2}{\pi}\frac{a}{\alpha^2 + a^2} \tag{15}$$

where :

$\xi(x)$ - white noise of intensity $s = 2\sigma_\varphi^2$

a - constant

With regard to Eq.(14) the Eq.(5) will be replaced by a system describing the three-dimensional Markov process

$$\left.\begin{aligned} v_1'(x) = & \quad v_2(x) \\ v_2'(x) = & \quad -\omega^2\psi^2 v_1(x) \ - \ \omega^2 v_3(x)v_1(x) \\ v_3'(x) = & \quad -av_3(x) + \xi(x) \ ; \ (v_3(x) = \varphi(x)) \end{aligned}\right\} \tag{16}$$

leading to the Fokker-Planck-Kolmogorov equation for the cross probability density of $p(v_1, v_2, v_3, x)$:

$$\frac{\partial p}{\partial x} = -\frac{\partial(v_2 p)}{\partial v_1} + \omega^2\psi^2\frac{\partial(v_1 p)}{\partial v_2} + \omega^2\frac{\partial(v_1 v_3 p)}{\partial v_2} + a\frac{\partial(v_3 p)}{\partial v_3} + \frac{\omega^2}{2}s\frac{\partial^2 p}{\partial v_3^2} \tag{17}$$

together with the initial conditions :

$$p(v_1, v_2, v_3, x)|_{x=0} = C\delta(v_1)\delta(v_2)\delta(v_3) \tag{18}$$

The same procedure as that used in the previous chapter will lead us with regard to Eq.(18) from Eq.(14) to the system of

$$\left.\begin{aligned} &\left.\begin{aligned} U_{1,0,0}' = & \quad U_{0,1,0} \\ U_{0,1,0}' = & \quad -\omega^2\psi^2 U_{1,0,0} - \omega^2 U_{1,0,1} \end{aligned}\right\} \quad l = 0 \\ &\left.\begin{aligned} U_{1,0,1}' = & \quad U_{0,1,1} - aU_{1,0,1} \\ U_{0,1,1}' = & \quad -\omega^2\psi^2 U_{1,0,1} - \omega^2\sigma_\varphi^2 U_{1,0,0} - aU_{0,1,1} \end{aligned}\right\} \quad l = 1 \end{aligned}\right\} \tag{19}$$

$$U_{j,k,l} = \iiint_{-\infty}^{\infty} p(v_1, v_2, v_3, x) v_1^j v_2^k v_3^l dv_1 dv_2 dv_3 \qquad (20)$$

In this process the system closing operation was performed in Eq.(19) by the introduction of the assumption of the quasi-Gaussian response. The individual moments in Eq.(19) have the following interpretations :

$U_{1,0,0}, U_{0,1,0}$ - mathematical mean values of displacement and its derivative (scale of normal force) respectively,

$U_{1,0,1}, U_{0,1,1}$ - cross correlation of the displacement and its derivative with the imperfections process.

We shall solve the system of Eq. (19) for the initial conditions

$$U_{1,0,0}|_{x=0} = K \; ; \; U_{1,0,1}|_{x=0} = 0 \qquad (21)$$

In this process the second initial condition results from the fully deterministic response in point $x = 0$. The remaining two initial conditions results from the Sommerfeld conditions. For the afore mentioned mathematical mean values we shall obtain

$$U_{1,0,0} = \frac{K}{4\xi}[(a+2\xi)e^{(-\frac{1}{2}a+\xi-i\eta)x} - (a-2\xi)e^{(-\frac{1}{2}a-\xi-i\eta)x}] \quad (22)$$

$$U_{0,1,0} = \frac{K}{4\xi}[(a+2\xi)\lambda_2 e^{(-\frac{1}{2}a+\xi-i\eta)x} - (a-2\xi)\lambda_4 e^{(-\frac{1}{2}a-\xi-i\eta)x}] \quad (23)$$

Eq.(22) is fully identical with the formula for the mathematical mean value of the displacement of $E\{v(x)\}$, computed by the integral spectral decomposition method, see Náprstek (1993). Both addends in Eqs. (22) and (23) have the same period and the damping differs the less the higher σ_0^2, i.e. the indeterminateness of the bar density. For $\sigma_0^2 \rightarrow 0$ the second addend approaches zero and the first changes into the solution of the classic deterministic problem. The derivative of the amplitude of both addens in Eq.(22) is identical and, consequently, the decrement of the amplitude of $U_{1,0,0}$, i.e. the rate of determinism of the displacement in the proximity of point $x = 0$, is very small. In this the investigated case differs from the case of viscous damping, when the greatest drop of amplitude occurs at the very proximity of the origin. In contradistinction to viscous damping, however, our case does not concern the dissipation of energy, concerning merely the successive tranformation of its form from the deterministic to the stochastic one. The drop of the rate of determinism of the normal force according to Eq.(23) corresponds approximately with the drop of the rate of determinism of the displacement according to Eq.(22). Only the phase shift of $U_{0,1,0}$ differs from that of $U_{1,0,0}$ and changes significantly with the dispersal σ_0^2, while for $\sigma_0^2 \rightarrow 0$ approaches $\pi/2$.

The correlation of the deviation and imperfection is described by the expression

$$U_{1,0,1} = \frac{-K\omega^2\sigma_0^2}{2\xi(a-2i\eta)}[e^{(-\frac{1}{2}a+\xi-i\eta)x} - e^{(-\frac{1}{2}a-\xi-i\eta)x}] \qquad (24)$$

The mutual relation of the displacement and imperfections in the origin equals zero. Its amplitude rises to a certain local maximum and then asymtotically approaches zero.

The solution given by Eqs.(22) or (23),(24) holds, if σ_0^2 is relatively small or, in other words, if the condition

$$0 < \sigma_0^2/\psi^2 < a^2/\omega^2 \qquad (25)$$

has been complied with. Under normal circumstances the condition (25) will be complied with, as a rule, if the spectral density (15) in point $\alpha = 0$ is not too "sharp". The relative dispersal of imperfections σ_0^2/ψ^2 is compared with the square of the inverted value of a certain velocity related with the length of the registrable correlation of the process $\varphi(x)$. The condition (24) is complied with, if the length of the propagating wave exceeds the length of this correlation. This divide recalls the critical frequencies in the continuum discretized by the FEM. except that in our case the divide does not consist in the crossing over the natural frequency of the subsystem, but in the crossing over the boundary of determinism of the parameters of the continuum. Beyond this boundary the problem has not more the character of a genuine random process problem. The cases in which the condition (25) is not complied with can be divided into three categories :

$$\sigma_0^2 < (\psi^2 + \frac{1}{4}\frac{a^2}{\omega^2})^2 \; ; \; \frac{a^2}{\omega^2} < 4\psi^2 \qquad (26)$$

$$\sigma_0^2 < (\psi^2 + \frac{1}{4}\frac{a^2}{\omega^2})^2 \; ; \; \frac{a^2}{\omega^2} \geq 4\psi^2 \qquad (27)$$

$$\sigma_0^2 \geq (\psi^2 + \frac{1}{4}\frac{a^2}{\omega^2})^2 \qquad (28)$$

Category (26) results in the solution characterized by beating in space, thus forming a certain transition area. The categories (27),(28) do not yield physically meaningful results. The solutions is either partly or entirely aperiodical which would mean that the excitation propagates at an infinitely high speed through such continuum. However, if $c^{-2}(x)$ drops to zero, it would mean a high probability of the state in which the imperfections will absorb entirely (on the dispersal level) the whole nominal value of corresponding parameter. This case is out of our analysis with regard to (2).

Therefore, we can conclude that physical meaning is yielded by the cases complying the condition (25) or maybe the cases from the transition region (26).

As it follows from the preceding considerations of the quasi-Gaussian response, the second order moments of the processes $v_1(x)$, $v_2(x)$ (i.e. $j + k = 2$) are the highest independent response moments - all that with the assumption that $l = 0$, i.e. without the links with imperfections.

Let us compile for the quantities $U_{2,0,0}, U_{1,1,0}, U_{0,2,0}$ a system similar to $U_{1,0,0}, U_{0,1,0}, U_{1,0,1}, U_{0,1,1}$ dealt with in the preceding chapter. The links with these four moments will originate expressing the moments $U_{2,0,1}, U_{1,1,1}$ with regard to the afore mentioned hypothesis of quasi-Gaussian character of the response. which will make the system closed and non-homogeneous.

After lengthy and laboursome considerations, while neglecting the higher order terms, we shall arrive at the expression for the dispersal of the displacement $v_1(x)$:

$$\begin{aligned} \sigma_v^2 = \; & (A + C + E) + (A - C + E)exp(-a + 2\xi)x - \\ & -2\sqrt{A^2 + B^2} \cdot cos(2\omega\psi x - \gamma_A) \cdot exp(-a + 2\xi)x - \\ & -2\sqrt{E^2 + F^2} \cdot cos((\omega\psi - \eta)x - \gamma_E) \cdot exp(-\tfrac{3}{2}a + \xi)x \\ & tg\gamma_A = B/A \quad ; \quad tg\gamma_E = F/E \end{aligned} \qquad (29)$$

In Eq.(29) the A, B, C, D, E, F are constants dependent on input parameters $E_0, \varrho_0, \omega, \sigma_0, a$. The first two terms are nonperiodical and represent a certain "trend" around which the remaining two terms, which are of the character of a linearly damped harmonic wave, oscillate.

The dispersal $\sigma_v^2(x)$ rises from zero in point $x = 0$, where the response process $v(x)$ is deterministic, and with rising x approaches asymtotically the sum of $A + C + E$. The damping of the first periodical term is relatively small and influences the

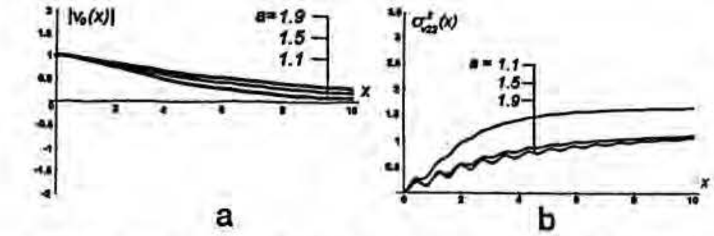

Figure 1: Amplitude of the mathematical mean value $|v_0(x)|$ and dispersal $|\sigma_v^2(x)|$ of the response ($\omega = 2, \psi = 2, \sigma_0 = 1$).

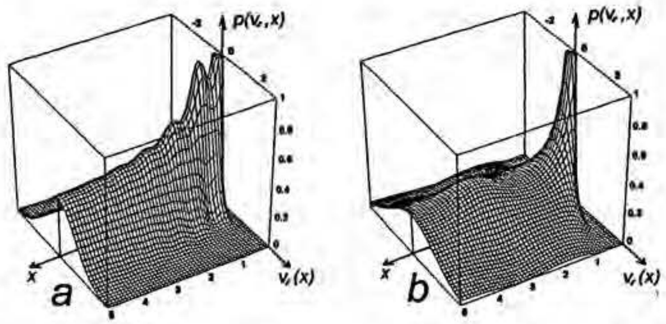

Figure 2: Probability density $p(v_\varepsilon, x)$ at various distances from the point of $x = 0$; a) $a = 1.1, \omega = 2, \psi = 2, \sigma_0 = 1$; b) $a = 1.1, \omega = 1, \psi = 1, \sigma_0 = 1$.

image of (29) much more significantly than the second periodic term the damping of which is much greater. The significance of the second term increases with the growing σ_0^2; in opposite case it has a tendency to disappear.

If we compare a few typical examples of the history of $U_{1,0,0}(x)$ and $\sigma_v^2(x)$ (see Fig.1), we can see that the drop of the deterministic component response with the growing distance from the excitation point is accompanied by the approximately the same increase of the stochastic component until the deterministic component practically disappears and the response becomes an almost homogeneous process with constant dispersal.

The successive transformation of energy form manifests itself on the probability density curve $p(v)$ for the individual points x. In point $x = 0$ this curve has the form of Dirac function which subsequently changes, in accordance with Eq. (29), into the "definite" Gauss curve (see Fig.2). This corresponds with the adopted assumtion of the pseudo-Gaussian character of the stochastic part of the response, if the imperfections are Gaussian.

CONCLUSIONS

In the propagation of harmonic wave motion in a continuum the material characteristic of which are burdened with random imperfections it is possible to observe many specific effects which have been identified by the analysis outlined in the preceding text.

The mathematical mean value of the response characterizing its deterministic part drops, slowly at first in the proximity of the excitation point. This is followed by a fast drop and finally by an asymptotic monotonous approach to zero for $x \to \infty$. In the corresponding intervals of x the dispersal increases from zero (fully deterministic process) first slowly, then very speedily, and finally approaches asymptotically the constant characterizing energy per unit volume induced into the bar. Typical is its alternating character while preserving the positive value. For x in the first two intervals, consequently, we can observe the response transformation from the deterministic into the stochastic part of the response with the preservation of the energy per unit volume.

The probability density curve changes in dependence on the distance from the excitation point from the Dirac function in point $x = 0$ (full deterministic response) over the curves with irregularly growing dispersal to the typical Gaussian curve of constant dispersal indicating that the process has become homogeneous.

Remarkable phenomenon is the existence of the upper limit of the excitation frequency and the lower limit of the length of the propagating wave which is comparable with the mean correlation length of imperfections (see an analogous effect in the FEM). There are also transition zones.

The mathematical model describing the imperfections as white noise is inapplicable, as it leads to results which either neglect the imperfections or are at variance with the energy equilibrium law. For the same reasons the small parameter method is inapplicable, too. It is necessary to apply at least the diffusional model which is characterized by the exponential correlation in space.

The conclusions, obtained in this study by means of the theory of Markov processes, are in full agreement with the results arrived at by the author by the method of integral spectral decomposition and the finite element method, i.e. by the methods based on an entirely different philosophy. The practical consequence of this and other author's papers is the recognition of the necessity of abandonment of the so far very widespread models of uncorrelated imperfections, as their application yields, particularly in infinite regions, in not only quantitatively, but also qualitatively false results.

REFERENCES

Deodasis G., Wall W., Shinozuka M. (1991), "Analysis of Two Dimensional Stochastic Systems by the Weighted Integral Method", in Proc. "Computational Stochastic Mechanics", Elsevier, London.

Nakagiri S. and Hisada T. (1985), "Stochastic Finite Element Method" (in Japanesse), Baifukan, Tokyo.

Náprstek J. (1993), "Propagation of Longitudinal Waves in a Beam with Randomly Variable Parameters", Euromech, 306, Mechanics of Contact Impact, Prague 1993.

Náprstek J. and Frýba L. (1994), "Interaction of a Long Beam on Stochastic Foundation with a Moving Random Load", in Proc. "5th Int.Conf. on Recent Advances in Structural Dynamics", Inst.of Sound and Vibrations, Southampton.

Pugachev V.S. and Sinitsyn I.N. (1987), "Stochastic Differential Systems-Analysis and Filtering", J.Wiley, Chichester.

Bolotin V.V. (1979), "Random Vibrations of Elastic Systems" (in Russian), Nauka, Moskva.